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CONVERGENCE OF RICCI FLOW ON R^2 TO PLANE

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Abstract. In this paper, we give a sufficient condition such that the Ricci flow in R^2 exists globally and the flow converges at $t = \infty$ to the flat metric on R^2.

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1. introduction

In this short note, we are interested in the long-term behavior on R^2 of conformally flat solutions to the Ricci flow equation on R^2. Recall here that the Ricci flow equation for the one-parameter family of metric $g(t)$ on R^2 is

$$\partial_t g = -Rg, \quad in\ R^2. \tag{1}$$

For these metrics $g(t)$, we take their forms as $g(x,t) = e^{u(x,t)} g_E$, where g_E is the standard Euclidean metric on R^2. Then the Ricci flow equation becomes

$$\partial_t e^u = \Delta u, \quad in \quad R^2, \tag{2}$$

where Δ is the standard Laplacian operator of the flat metric g_E in R^2. The long-term existence of solutions of (1) or (6) has been studied in [3], , where it is shown that

Theorem 1. *The solutions to (1) with initial metric $g(0) = e^{u_0} g_E$ exist for all $t \geq 0$ if and only if*

$$\int_{R^2} e^{u_0} dx = \infty. \tag{3}$$

The global behavior of the Ricci flow has been studied in [14]. To state one of her result, we recall two concepts of the metric $g = g(t)$. One is below.

Definition 2. *The* aperture *of the metric g on R^2 is defined as*

$$A(g) = \frac{1}{2\pi} \lim_{r\to\infty} \frac{L(\partial B_r)}{r}.$$

Here B_r denotes the geodesic ball (or disc) of radius r and $L(\partial B_r)$ is the length of the boundary of ∂B_r.

The other is the Cheeger-Gromov convergence of the Ricci flow.

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Definition 3. *The Ricci flow $g(t)$ is said to have modified subsequence convergence, if there exists a 1-parameter family of diffeomorphisms $\{\phi(t)\}_{t_j\geq 0}$ such that for any sequence $t_j \to \infty$, there exists a subsequence (denoted again by t_j) such that the sequence $\phi_(t_j)^* g(t_j)$ converges uniformly on every compact set as $t_j \to \infty$.*

Then we have the following result of L.F. Wu [14].

Theorem 4. *Let $g(t) = e^{u(t)} g_E$ be a solution to (1.1) such that $g(0) = e^{u_0} g_E$ is a complete metric with bounded curvature and $\nabla u_0|$ is uniformly bounded on R^2. Then the Ricci flow has modified subsequence convergence as $t_j \to \infty$ with the limiting metric g_∞ being complete metric on R^2; furthermore, the limiting metric is flat if $A(g(0) > 0$.*

We point out that the diffeomorphisms $\phi(t_j)$ used in Theorem 4 are of the special form

$$\phi(t)(a, b) = (e^{\frac{-u(x_0,t)}{2}} a, e^{\frac{-u(x_0,t)}{2}} b) = (x_1, x_2) = x,$$

where $x_0 = (0, 0)$. The important fact for these diffefeomorphisms is that

$$|\nabla_{g(t)} f(x, t)| = |\nabla_{\phi(t)^* g(t)} f((a, b), t)|.$$

for any smooth function f and $x = \phi(t)(a, b)$.

In the interesting paper [8], which motivates our work here, the authors have proved the following.

Theorem 5. *Suppose $g_0 = e^{u_0} g_E$ has bounded curvature and u_0 is a bounded smooth function on R^2. Then the Ricci flow $\partial_t g = -Rg$ exists for all $t \geq 0$ and has modified subsequence convergence to the flat metric in the C^k topology of metrics on compact domains in R^2 for each $k \geq 2$.*

There is another formulation in dimension two. Since every complete Riemannian manifold of dimension two is a one dimension Kähler manifold, we can use the Kähler-Ricci flow formulation of the Ricci flow on R^2. We shall consider the Ricci flow (1) as the Kähler-Ricci flow by setting

$$g_{i\bar{j}} = g_{0i\bar{j}} + \partial_i \partial_{\bar{j}} \phi,$$

where $\phi = \phi(t)$ is the Kähler potential of the metric $g(t)$ relative to the metric g_0. Note that

$$g(0)_{i\bar{j}} = g_{0i\bar{j}} + \partial_i \partial_{\bar{j}} \phi_0,$$

In this situation, the Ricci flow can be written as

$$\partial_t \phi = \log \frac{g_{01\bar{1}} + \phi_{1\bar{1}}}{g_{01\bar{1}}} - f_0, \quad \phi(0) = \phi_0, \tag{4}$$

where f_0 is the potential function of the metric g_0 in the sense that $R(g_0) = \Delta_{g_0} f_0$ in R^2. Here $\Delta_{g_0} = g_0^{1\bar{1}} \partial_1 \partial_{\bar{1}}$ in R^2, which is the normalized Laplacian in Kähler geometry. Such a potential function has been introduced by

R.Hamilton in [7]. We remark that the initial data for the evolution equation (4) is $\phi(0)$ which is non-trivial. For the equivalent of these two flows, one may see [1].

Our result is below.

Theorem 6. *Suppose $g_0 = e^{u_0}g_E$ has bounded curvature R_0 with (3) and f_0 is a bounded smooth function on R^2 such that $\Delta_{g_0}f_0 = R_0$. Then the Ricci flow $\partial_t g = -Rg$ with the initial metric g_0 exists for all $t \geq 0$ and has modified subsequence convergence to the flat metric in the C^k topology of metrics on compact domains in R^2 for each $k \geq 2$.*

We remark that because of the assumption about the potential function f_0, the initial metric g_0 is far from the cigar metric [9]. Here is the idea of the proof. We shall show that the limit f_∞ of $f(t_j)$ is a constant function. Because of Theorem 4, we need only show that $R(g_\infty) = \Delta_{g_\infty}f_\infty = 0$. The proof of Theorem 6 will be given in section 3.

2. MAXIMUM PRINCIPLE AND THE EQUIVALENCE OF FLOWS (4) AND (1) IN DIMENSION TWO

Fist we recall the maximum principle for the Ricci flow with bounded curvature. Given the Ricci flow $g(t)$ on R^2 with bounded curvature, we have the following well-known maximum principle.

Lemma 7. *Fix any $T > 0$. If $w(x,t)$ is a bounded smooth solution to the heat equation*

$$\partial_t w = \Delta_{g(t)}w, \quad R^2 \times (0,T]$$

with the bounded initial data $w(x,0)$, then $|w(x,t)| \leq \sup_{R^2}|w(x,0)|$ for all $t \in (0,T]$.

We now consider the equivalence of the flows (4) and (1) in dimension two. We use the argument in [1] (see Lemma 4.1 there). If $g(t)$ is the Ricci flow in (1), we define

$$u(x,t) = \int_0^t \log\frac{g_{1\bar{1}}(x,s)}{g_{1\bar{1}}(x,0)}ds - tf(0)$$

and

$$S_{1\bar{1}}(x,t) = g_{1\bar{1}}(x,t) - g_{1\bar{1}}(x,0) - u_{1\bar{1}}(x,t).$$

Then by direct computation we have

$$\frac{dS_{1\bar{1}}(x,t)}{dt} = 0, \quad S_{1\bar{1}}(x,0) = 0.$$

Hence $S_{1\bar{1}}(x,t) = 0$ for all $t > 0$ and

$$g_{1\bar{1}}(x,t) = g_{1\bar{1}}(x,0) + u_{1\bar{1}}(x,t).$$

If $u = u(x,t)$ is a solution to (4), then it is clear that

$$g_{1\bar{1}}(x,t) = g_{1\bar{1}}(x,0) + u_{1\bar{1}}(x,t)$$

satisfies (1).

3. PROOF OF THEOREM 6

The idea of the proof of Theorem 6 is similar to the argument in [2] and [8], see also [9].

Let
$$f = -\partial_t \phi.$$
Then, taking the time derivative of (4), we have
$$\partial_t f = \Delta_g f, \quad f(0) = -\partial_t \phi(0) = f_0. \tag{5}$$
By Lemma 7 we know that f is uniformly bounded in R^2. The important fact for us is that
$$\Delta_g f = R. \tag{6}$$
See [9] for a proof of this. if f_0 has some decay at space infinity, one can can the same decay by the argument of Dai-Ma [4].

It is well-known that R is uniformly bounded in any finite interval and $|f_t|$ and $|\nabla f|^2$ are bounded for each $t \geq 0$ (via the use of $f(x,t) = f(x,0) + int_0^t R(x,s)ds$).

Our next task is to obtain a better control on $|\nabla f|$ as $t \to \infty$. To get this, we let
$$F(x,t) = t|\nabla f|^2 + f^2.$$
Then we have
$$\partial_t F \leq \Delta_g F, \quad in\ R^2.$$
Using the maximum principle (Lemma 7), we know that
$$\sup_{x \in R^2} |\nabla f(x,t)|^2 \leq \frac{C}{1+t}$$
for some uniform constant $C > 0$. Once we have this bound, we can follow the argument in Lemmata 8,9,and 10 in [8] to conclude that the curvature bounds that there are uniform constants C_k, for any $k \geq 1$, such that
$$\sup_{R^2} |\nabla^k R(x,t)|^2 \leq \frac{C_k}{(1+t)^{k+2}}, \ t > 0. \tag{7}$$

We are now ready to complete the proof of Theorem 6 *Proof of Theorem 6.* We shall use the modified convergence sequence $g(t_j)$ in Theorem 4. We need only show that the limiting metric has flat curvature and this will be obtained by showing that the limiting function f_∞ of $f(x,t_j)$ is constant. Since $f(x,t)$ is uniformly bounded by a constant $K > 0$ on $R^2 \times [0,\infty)$, for the fixed $x_0 = (0,0) \in R^2$ and for any sequence $t_j \to \infty$, there exists a subsequence, still denoted by t_j, such that $c = \lim f(x_0,t_j)$ exists. By the construction of the metrics $g(t_j)$, for any compact subset $K \subset R^2$, the limiting metric g_∞ is equivalent to any $\phi(t_j)^* g(t_j)$ for every large t; that is, there is a uniform constant $C = C(K) > 0$ such that
$$d_t(x,x_0) \leq C d_{g_\infty}(x,x_0)$$

for every $x \in K$, where $d_t(x, x_0)$ is the distance between x and x_0 in $\phi(t)^*g(t)$ and $d_{g_\infty}(x, x_0)$ is the distance of the limiting metric.. For $x \in K$, we can establish (for all $t > 1$),

$$|f(x,t) - f(x_0,t)| \leq d_t(x,x_0) \sup_{x \in K} |\nabla f(x,t)| \leq \frac{C d_{g_\infty}(x,x_0)}{1+t},$$

where we have used the fact that for $x = \phi(t)(a,b)$,

$$|\nabla_{g(t)} f(x,t)| = |\nabla_{\phi(t)^*g(t)} f((a,b),t)|,$$

which is uniformly bounded. It follows that $f(x, t_j)$ is also convergent to c, which is $f_\infty(x) = c$ as $t_j \to \infty$, and then $\partial_1 \partial_{\bar{1}} f(x, t_j) \to 0$. Then we have $\Delta_{g_\infty} f_\infty = 0 = R_\infty$ and then $\phi(t_j)^* g(t_j) \to g_\infty$ locally in C^2 with g_∞ of flat curvature. The C^k-convergence of $\phi(t_j)^* g(t_j)$ to this flat limit then follows from the previous curvature estimates obtained in (7) (see Lemmata 8,9, and 10 in [8]). This completes the proof of Theorem 6.

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